The Physics of Deterministic Chaos : Implications for Global Climate Model Predictions

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ABSTRACT

Finite precision computer realizations of numerical weather prediction (NWP) and climate models are sensitively dependent on initial conditions and give chaotic solutions, now identified as deterministic chaos, an area of intensive research in all branches of science and other areas of human interest. Deterministic chaos is a consequence of the following inherent errors in numerical computations. (a) The continuum dynamical system such as atmospheric flows is computed as a discrete dynamical system with implicit assumption of subgrid-scale homogeneity. (b) Model approximations and assumptions. (c) Binary number representation in digital computers precludes exact number representation at the data input stage itself. (d) Round-off error of finite precision computer arithmetic magnifies exponentially with time the above uncertainties and gives unrealistic solutions. The physics of deterministic chaos is not yet identified. In this paper, a summary of a recently developed nondeterministic cell dynamical system model for deterministic chaos is presented. The model predicts approximate round-off error doubling on an average for each iteration with propagation of round-off error to mainstream computation within 100 iterations in single precision computations. Round-off error will propagate to mainstream computation and give unrealistic solutions in numerical weather prediction (NWP) and climate models which incorporate thousands of iterations in long-term numerical integration schemes.

INTRODUCTION

'Chaos' means disorder or lack of predictability. Deterministic chaos is the name given to the sensitive dependence on initial conditions of finite precision computer realizations of nonlinear mathematical models of dynamical systems. A dynamical system is a system that changes it behaviour with time, e.g., weather, climate, stock market price fluctuations, spread of infectious diseases in a population, heart rhythms, electrical activity of the human brain, etc. Though the governing equations are 'deterministic', i.e., precisely formulated and written down, the numerical solutions are chaotic because of sensitive dependence on initial conditions and therefore named 'deterministic chaos'. Deterministic chaos precludes long-term predictability. Sensitive dependence on initial conditions was identified nearly a century ago by Poincarè in his study of the three body problem. Advent of computers and graphical display facilities in the late nineteen fifties facilitated numerical solutions and in 1963 Lorenz (1) identified sensitive dependence on initial conditions in a simple mathematical model of atmospheric flows. Ruelle and Takens (2) were the first to identify the similarity between deterministic chaos and turbulence in fluid flows; turbulence is as yet an unresolved problem in fluid dynamics. Investigations by scientists and mathematicians revealed surprising order underlying deterministic chaos, such as the universal Feigenbaum's constants (3) **a** and **d** which characterise the universal period doubling route to chaos, where, transition to chaos occurs by successive period doublings in the computed result. Since 1980 'Chaos Science' is an intensive area of research in all branches of science and other areas of human interest such as economics, politics and art (4). The delicate and beautiful patterns of deterministics chaos generated by iterative solutions of simple nonlinear equations as seen on computer colour graphics has fascinated people in general and accounts for the widespread interest in this field. The physics of deterministic chaos is not yet identified. Simple nonlinear equations give chaotic i.e., random solutions. Random chance, i.e., unpredictable behaviour governs many natural phenomena such as weather and climate. Investigations in Chaos Science may help simulate complex, apparently unpredictable behaviour by means of simple equations and help identify a 'Theory of Everything' (TOE).

In this paper, a brief summary and current status of deterministic chaos with special reference to weather and climate prediction, followed by, a summary of a nondeterministic cell dynamical system model for deterministic chaos in computer realizations of dynamical systems (5) is presented.

DETERMINISTIC CHAOS : CURRENT STATUS

Traditionally, mathematical models of dynamical systems are formulated using Newtonian continuum dynamics where it is assumed that the rate of change dX/dt of variable X with time t is continuous over infinitesimally small intervals of time dt. A dynamical system is characterised by M variables X_i, i = 1,...M. The governing equations for the time evolution of the dynamical system are written as

$$\frac{dX_i}{dt} = F_i \; (X_i, \; i = 1,M) \tag{1}$$

The governing equations, in general, are nonlinear i.e., without analytical solutions. Numerical solutions are then obtained as successive iterations such as

$$X_{n+1} = X_n + \left[\frac{dX}{dt} \right] dt \qquad dt \approx 0 \tag{2}$$

or

$$X_{n+1} = F \; (X_n) \tag{3}$$

where the value X_{n+1} of the variable X at the (n+1)th instant is a function F of X_n with implicit error feedback loop. The following errors are inherent to finite precision computer realizations of continuum dynamical systems such as Eq.(3) : (a) Continuum dynamical system (Eq.2) is computed as discrete dynamical system such that

$$X_{n+1} = X_n + \left[\frac{\Delta X}{\Delta t} \right]_n \Delta t \tag{4}$$

$$\Delta t \gg 0$$

with implicit assumption of sub-grid scale homogeneity (b) Binary computer arithmetic precludes exact number representation at the data input stage itself (c) Errors of model approximations and assumptions. Round-off error of finite precision computer arithmetic magnifies exponentially with time the above errors (a) to (c) and gives chaotic solutions, i.e., deterministic chaos. Deterministic chaos in time evolution has been investigated extensively as compared to spatial evolution of the dynamical system. Since the spatial evolution is a function of time, the spatial pattern is also expected to exhibit deterministic chaos, i.e., sensitive dependence on initial conditions. Sensitive dependence on initial conditions implies long-term spatiotemporal correlations in computed solutions.

Computed solutions are plotted in the phase space (an abstraction) defined by M co-ordinates which represent the M variables of the dynamical system. The values of M variables at any instant are plotted as a point in the M-dimensional phase space, e.g., the u,v,w momenta of an air parcel along the x,y and z directions respectively will have a 6-dimensional phase space. The line joining the successive points in phase space gives the trajectory of the dynamical system. The trajectory traces a 'strange attractor' (another abstraction), so-called because of its strange convoluted shape is the final destination of all possible trajectories. Two initially close points diverge exponentially with time though still within the strange attractor domain. The strange attractor has selfsimilar fractal geometry. Self-similarity implies identical geometrical shape at all scales of magnification, i.e., the sub-units resemble the whole. The word 'fractal' coined by Mandelbrot (6) in 1977 implies broken (fractional) Euclidean structure whose fractal dimension D is given by $D = d\ln M/d\ln R$ where M is the mass contained within a distance R from a point in the extended object. A constant value for D implies uniform stretching on a logarithmic scale. Objects in nature, in general possess a multifractal structure, i.e., the fractal dimension D is different for different length scales R. The selfsimilar fractal spatial pattern of dynamical systems evolve by selfsimilar fluctuations on all time scales. Selfsimilar fractal geometry to the spatial pattern implies long-range spatial correlations. The power spectrum of temporal fluctuations is broadband, and follows the inverse power law form $1/f^B$ where f is the frequency and B the exponent. Inverse power law form for power spectra imply long-range temporal correlations or persistence, i.e., memory. Deterministic chaos in computed dynamical systems is therefore characterised by long-range spatiotemporal correlations. Such long-range spatiotemporal correlations are ubiquitous to dynamical systems in nature and is recently identified as signatures of self-organized criticality (7). Atmospheric flows exhibit long-range spatiotemporal correlations as manifested in the selfsimilar fractal geometry to the global cloud cover pattern concomitant with inverse power law form for power spectra of temporal fluctuations documented by Lovejoy and Schertzer (8).

The physics of deterministic chaos, i.e., self-organized criticality in real world and computed model dynamical systems is not yet identified. The fidelity of computed model solutions is questionable in the absence of analytical (true) solutions (9). Computed model predictions should be accepted with caution. Alternatives for more realistic prediction are statistical models such as the 16-parameter long-range monsoon prediction model of Gowarikar et al (10) based on well documented long-range spatiotemporal correlations, namely, self-organized criticality in atmospheric flows.

Realistic prediction of dynamical evolution of real world systems such as atmospheric flows, therefore require alternative concepts for physical laws, formulation of structurally stable governing equations which are stable to small perturbations and robust computational techniques which do not incorporate error feedback as in the case of numerical integration schemes. It is therefore required to formulate simple (algebraic) governing equations with analytical solutions or solutions which do not require numerical integration. In this paper a non deterministic cell dynamical system model for deterministic chaos in computer realizations of dynamical system (5) is summarised. The model predicts a approximate round-off error doubling on an average for each iteration. Round-off error will propagate into the mainstream computation and give unrealistic solutions in NWP and climate models which incorporate thousands of iterations in long-term numerical integration schemes.

MODEL CONCEPTS

In summary, deterministic chaos is a direct result of round-off error growth in iterative computations (Eq.3), i.e., long-term numerical integration schemes. Round-off error growth is visualised in Fig. 1 and explained in the following. In single precision computations (computer) the precision is 10^{-7} or the round-off error is 10^{-7}. Computer precision is analogous to yardstick length dR in length measurement. Two points separated by a distance les than yardstick length dR cannot be distinguished as separate. In the following discussions computer precision is treated as analogous to yardstick length in length measurement. One unit of length measurement of yardstick length dR has the following two inherent uncertainities (a) Lengths less than dR will be measured as equal to dR. (b) Lengths less than 2dR will also be measured as equal to dR. The uncertainty domain associated with unit measurement of yardstick length dR can be represented by a circle $OR_2R_1'R_2$ of radius OR_1 equal to dR (Fig. 1). The precision decreases or the yardstick length R increases with successive iterations (Eq.3). In the following discussions dR or r refers to the precision inherent to the computational system comprising of the digital computer and the input uncertainities of the model dynamical system. The increased imprecision with successive iterations is represented by increasing yardstick length R. The computational domain, namely, the strange attractor, is expressed as the product WR of the number of units of computation W of yardstick length (precision) R. The spatial integration of microscopic round-off error domains $OR_2R_1'R_2$ of macroscale lengths R gives the computed domain (Fig. 1). The growth of uncertainity domain with successive iterations to give the macroscale strange attractor pattern is analogous to formation of large eddy structures as envelopes of enclosed microscope scale eddy fluctuation domains (11,12). The computed strange attractor pattern can be visualised as the envelope of enclosed microscopic scale uncertainity domains $OR_2R_1'R_2$. w_* units of computation of yardstick length dR may be represented as a rectangle of sides w_* and dR. The spatial integration of such round-off error domains results in W units of computation of decreased precision, i.e., increased yardstick length R. W units of computation of yardstick length R may be expressed as a function of higher precision computational domain w_*dR as (5)

$$W^2 = \frac{\pi}{2}\frac{dR}{R}w_*^2 \qquad (5)$$

w_* units of computation of yardstick length dR forms the higher precision earlier stage computation for the next stage.

Therefore Eq.(5) may be written as

$$W_{n+1}^2 = \frac{2}{\pi} \frac{R_{n+1}}{R_n} W_n^2 \qquad (6)$$

Starting (n=1) with one unit of computation (W_1) of unit yardstick length (R_1=1), the uncertainity $(dR)_1$ in the computation is equal to the number of units of computation, i.e., $(dR)_1 = W_1 = 1$, since one unit of computation generates one unit of uncertainty. At the end of the first stage of computation the uncertainity or yardstick length increase to R_2 [$=R_1+(dR)_1=2$] and is equivalent to W_2(=1.284) units of computation from Eq.6. The successive values of W and R are found to follow the Fibonacci mathematical number series. A polar diagram (Fig. 2) of the successive values of yardstick length R or the number of units of computation W traces a logarithmic spiral with Fibonacci winding number and the quasiperiodic Penrose tiling for the internal structure. Since the computed domain (strange attractor) is resolved as the product WR of the number of units of computation W of yardstick length R, the computed domain can also be resolved into the quasiperiodic Penrose tiling pattern. The overall trajectory of the dynamical system traces the logarithmic spiral $R = re^{b\Theta}$ where b = tanα, α being the crossing angle (Fig 2) for small angular turning Θ

$$\tan\alpha = \alpha = \frac{dR}{R} = \frac{1}{\tau}$$

Therefore the equation for the logarithmic spiral which quantifies the exponential increase of uncertainity in initial conditions is given as

$$R = r\, e^{\Theta/\tau}$$

An initial uncertainity, i.e., yardstick length r, grows to 1.855r for unit angular turning , i.e., $\pi/5$ radians for each length step growth with Fibonacci winding number. The uncertainity therefore doubles on average for each iteration. Round-off error will propagate into mainstream computation within 100 iterations and give unrealistic solutions in conventional numerical weather prediction and climate models which incorporate thousands of iterations in long-term numerical integration schemes.

The model also enables to show that the power spectra of chaotic dynamical systems follow the universal inverse power law form of the statistical normal distribution (5) thereby illustrating the universality underlying the round-off error growth dynamics. The computed strange attractor is therefore a mathematical artefact of the universal process of round-off error growth in computed dynamical systems. Earlier numerical experiments (13) had shown that the round-off error doubles for each iteration.

CONCLUSIONS

The important conclusion of the present study is that round-off error will propagate into mainstream computation and give unrealistic solutions in numerical weather prediction (NWP) and climate models which incorporate thousands of iterations in long-term numerical integration schemes.

Realistic prediction of weather and climate require alternative concepts for physical laws and formulation of structurally stable governing equation, i.e., stable to small perturbations and robust computational techniques which do not have error feedback to mainstream computation. It is now realised that realistic simulation requires formulation of simple (algebraic) governing equations with analytical solutions or solutions which do not require numerical integrations.

It has been possible to simulate realistically the observed self-organised criticality in atmospheric flows by a recently developed nondeterministic cell dynamical system model for atmospheric flows (11,12).

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